

**GAUGE THEORY MODEL <sup>a</sup>**  
**– Quark Dynamics and Anti-ferromagnets –**

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Two-dimensional QED with N-flavor fermions serves as a model of quark dynamics in QCD as well as an effective theory of an anti-ferromagnetic spin chain. It is reduced to N-degree quantum mechanics in which a potential is self-consistently determined by the Schrödinger equation itself.

**QCD and antiferromagnets**

There are three reasons to investigate two-dimensional QED with massive fermions: <sup>1</sup>

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^N \bar{\psi}_a \left\{ \gamma^\mu (i\partial_\mu - eA_\mu) - m_a \right\} \psi_a \quad . \quad (1)$$

First of all, it carries many features essential in QCD; confinement, chiral condensates, and  $\theta$  vacuum. One can evaluate various physical quantities such as chiral condensates, Polyakov loop, and string tension at zero and finite temperature with arbitrary fermion masses to explore QCD physics.

Secondly QED is an effective theory of spin systems.<sup>2</sup> A  $s = \frac{1}{2}$  anti-ferromagnetic spin chain

$$H = J \sum \vec{S}_n \cdot \vec{S}_{n+1} \quad (S = \frac{1}{2}, J > 0) \quad (2)$$

is equivalent to two flavor massless QED<sub>2</sub> in a uniform charge background in the strong coupling limit. Similarly a spin ladder system is equivalent to coupled QED<sub>2</sub>. Their correlation functions can be evaluated systematically. Physics of two-dimensional anti-ferromagnets is the core to understand the high  $T_c$  superconductivity, for which QED description provides an indispensable tool.

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Thirdly there is significant development in technology. We show that a field theory problem is reduced to a quantum mechanics problem of finite degrees of freedom, which can be solved numerically on work stations. Technically this method is much simpler and easier to handle than the lattice gauge theory or light front method.

### Reduction to quantum mechanics

Consider the model (1) defined on a circle  $S^1$  with a circumference  $L$ . With periodic and anti-periodic boundary conditions imposed on bosonic and fermionic fields, the model is mathematically equivalent to a theory defined on a line ( $R^1$ ) at finite temperature. Hence various physical quantities at  $T \neq 0$  on  $R^1$  are obtained from the corresponding ones at  $T = 0$  on  $S^1$  by substituting  $L$  by  $T^{-1}$ .

$\psi_a$  is bosonized on a circle. It is expressed in terms of zero modes  $q_a$  and oscillatory mode  $\phi_a(x)$ . The only physical degree of freedom associated with gauge fields is the Wilson line phase  $\Theta_W$  along the circle.

When fermions are massless, zero modes ( $q_a, \Theta_W$ ) decouple from oscillatory modes  $\phi_a$ . The latter consists of 1 massive boson and  $N - 1$  massless bosons. The model is exactly solvable.

Fermion masses provide nontrivial interactions among zero and oscillatory modes. All boson fields become massive. When fermion masses are degenerate,  $m_a = m$ , we have 1 heavy boson with mass  $\mu_1$  and  $N - 1$  lighter bosons with mass  $\mu_2$ . The vacuum wave function is written as  $f(p_W, \varphi_1, \dots, \varphi_{N-1}; \theta)$ .  $\theta$  is the vacuum angle parameter of the theory.  $f(p_W, \varphi)$  must satisfy

$$\begin{aligned} \{K + V\} f &= \epsilon f \\ K &= -\frac{N^2}{4\pi^2} \frac{\partial^2}{\partial p_W^2} - (N-1) \left\{ \sum_{a=1}^{N-1} \frac{\partial^2}{\partial \varphi_a^2} - \frac{2}{N-1} \sum_{a < b}^{N-1} \frac{\partial^2}{\partial \varphi_a \partial \varphi_b} \right\} \\ V(p_W, \varphi) &= \frac{(\mu L p_W)^2}{4} - \frac{NmL\bar{B}}{\pi} \sum_{a=1}^N \cos\left(\varphi_a - \frac{2\pi p_W}{N}\right) \end{aligned} \quad (3)$$

where  $\sum_{a=1}^N \varphi_a = \theta$ ,  $\mu^2 = Ne^2/\pi$  and  $\bar{B} = B(\mu_1 L)^{1/N} B(\mu_2 L)^{(N-1)/N}$ .  $B(z)$  is given by  $B(z) = (z/4\pi)e^{\gamma+(\pi/z)} \exp\left\{-2 \int_0^\infty dx/(e^{z \cosh x} - 1)\right\}$ .

The boson masses are determined by

$$\mu_1^2 = \mu^2 + \mu_2^2 \quad , \quad \mu_2^2 = \frac{8\pi m \bar{B}}{L} \langle \cos\left(\varphi_a - \frac{2\pi p_W}{N}\right) \rangle_f \quad (4)$$

where  $\langle \rangle_f$  denotes the  $f$  average. (3) and (4) are solved simultaneously. Schematically  $V(p_W, \varphi) \rightarrow f(p_W, \varphi) \rightarrow \mu_\alpha \rightarrow V(p_W, \varphi)$ . This is a Schrödinger problem in which the potential has to be reproduced by the equation itself.

## Quark dynamics

Chiral condensates are given by

$$\langle \bar{\psi}_a \psi_a \rangle_\theta = -\frac{2\bar{B}}{L} \langle \cos \left( \varphi_a - \frac{2\pi p_W}{N} \right) \rangle_f . \quad (5)$$

The string tension  $\sigma$  between two external sources, one with charge  $+q$  and the other with  $-q$ , is

$$\sigma = Nm \left\{ \langle \bar{\psi} \psi \rangle_{\theta_{\text{eff}}} - \langle \bar{\psi} \psi \rangle_\theta \right\} , \quad \theta_{\text{eff}} = \theta - \frac{2\pi q}{e} . \quad (6)$$

These quantities are evaluated numerically with arbitrary values for  $L = T^{-1}$ ,  $m$ , and  $\theta$ . At  $T = 0$  and for  $m \ll \mu$ ,

$$\frac{\sigma}{\mu^2} = -\frac{N}{2\pi} \left( 2e^\gamma \frac{m}{\mu} \right)^{\frac{2N}{N+1}} \left\{ \left( \cos \frac{\bar{\theta}_{\text{eff}}}{N} \right)^{\frac{2N}{N+1}} - \left( \cos \frac{\bar{\theta}}{N} \right)^{\frac{2N}{N+1}} \right\} \quad (7)$$

where  $\bar{\theta} = \theta - 2\pi[(\theta + \pi)/2\pi]$ . The  $q$  dependence of the string tension at various temperature is displayed in fig. 1.

The following conclusions are obtained.

1. At  $T = 0$  the chiral condensate is not analytic in  $m$ .<sup>3</sup>
2. At  $T = 0$ , there appears a cusp singularity at  $\theta = \pi$ .
3. Sufficiently large asymmetry in fermion masses removes the cusp singularity at  $\theta = \pi$ .
4. Witten's picture<sup>4</sup> of chiral dynamics in QCD<sub>4</sub> is reproduced in QED<sub>2</sub>.
5. The string tension vanishes for an integer  $q/e$ .<sup>5</sup>
6. The string tension is non-vanishing only when  $m \langle \bar{\psi} \psi \rangle$  has non-trivial  $\theta$  dependence.
7. The chiral condensate increases as a fermion mass becomes very large:  $\langle \bar{\psi} \psi \rangle_{T=0} \sim -(e^{2\gamma}/\pi) m$  for  $m \gg \mu$ .
8. However, a contribution of a heavy fermion to the string tension becomes negligible as its mass becomes large.
9. At  $\theta = \pi$  a discontinuity in chiral condensates develops at a critical fermion mass  $m_c$ .

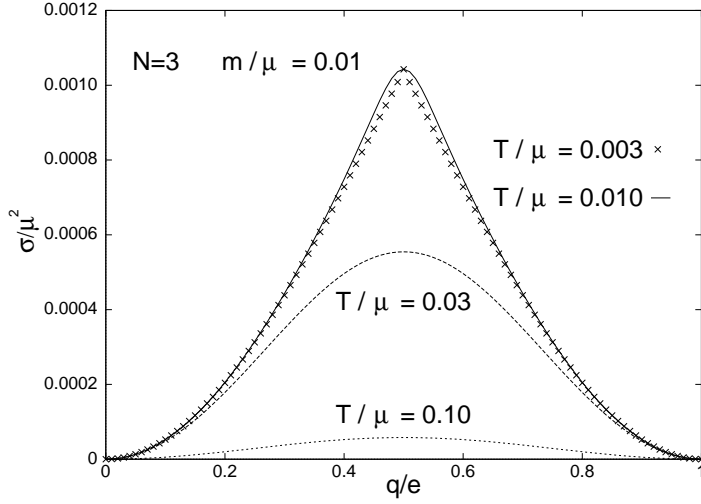


Figure 1: The charge,  $q$ , dependence of the string tension in the  $N = 3$  model with  $m/\mu = .01$  at  $\theta = 0$  at various temperature  $T$ . At  $T = 0$  a cusp singularity develops at  $q = \frac{1}{2}e$ .

### Gauge theory of anti-ferromagnetic spin chains

A  $s = \frac{1}{2}$  spin chain, (2), is equivalent to QED<sub>2</sub>. The derivation goes as follows.

Write  $\vec{S}_n = c_n^\dagger \frac{1}{2} \vec{\sigma} c_n$  where  $c_{n\alpha}$  is an annihilation operator of an electron at site  $n$  with spin  $\alpha$ . The Hamiltonian (2) is transformed to the Lagrangian

$$L^{(1)} = \sum \left\{ i c_n^\dagger \dot{c}_n + \phi_n (c_n^\dagger c_n - 1) - \frac{J}{2} (\chi_n^* \chi_n - \chi_n c_n^\dagger c_{n+1} - \chi_n^* c_{n+1}^\dagger c_n) \right\}. \quad (8)$$

$\chi_n$  is an link variable, defined on the link connecting sites  $n$  and  $n + 1$ . The Lagrangian  $L^{(1)}$  has local  $U(1)$  gauge invariance as well.

In a spin chain, the magnitude of  $\chi_n$  is almost frozen, ie. the effective potential for  $|\chi_n| = \chi$  has a sharp minimum at  $\chi = 1/\pi$ , the curvature there being proportional to the lattice volume. To a very good approximation, one can write  $\chi_n = (i/\pi) e^{ia_0 A_n}$  where  $a_0$  is the lattice spacing.  $\phi_n$  and  $A_n$  are the time and space components of a  $U(1)$  gauge field  $A_\mu(x)$ .

In an anti-ferromagnetic spin chain two adjacent sites form one block. Each block contains four electron states. The coupling of the gauge field  $A_\mu$  in  $L^{(1)}$  is spin-blind. Therefore, an electron spin becomes flavor in the continuum Dirac field, whereas an even-odd site index becomes a spin index.

$$\begin{array}{ccc}
c_{a\alpha} & \Longleftrightarrow & \psi_a^{(\alpha)} \\
\alpha : & \text{spin} & \text{flavor} \\
a : & \text{even-odd} & \text{spin}
\end{array} \tag{9}$$

In the continuum limit  $L^{(1)}$  becomes

$$\mathcal{L}^{(2)} = -\frac{1}{4e^2} F_{\mu\nu}^2 + \sum_{\alpha} \bar{\psi}^{(\alpha)} i\gamma^{\mu} (\partial_{\mu} - iA_{\mu}) \psi^{(\alpha)} + \frac{1}{a_0} A_0 \quad . \tag{10}$$

We have added the Maxwell term with the understanding that the limit  $e^2 \rightarrow \infty$  is taken at the end. Light velocity in  $\mathcal{L}^{(2)}$  is given by  $c = a_0 J/\pi$ .

A  $s = \frac{1}{2}$  spin chain is equivalent to 2-flavor massless QED<sub>2</sub> in the strong coupling limit in a uniform background charge. After bosonization 2-flavor QED<sub>2</sub> contains two bosons, one with  $(\text{mass})^2 = 2e^2/\pi$  and the other with a vanishing mass. In the  $e^2 \rightarrow \infty$  limit the former decouples. The latter, the massless boson, is the gapless mode known in the Bethe ansatz solution. It controls the long-range behavior of various correlation functions.

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